NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 HSC Course Assessment Task 3 - Trial

18 June 2021

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.

SECTION 1

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTIOIN 2

- Commence each new section on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete writing.

STUDENT NUMBER .... No.of booklets used ....

Class (please tick)

◯ 12 MM4A Mr Lin ◯ 12MM4B Mr Umakanthan ◯ 12MM4C Mr Berry

Marker's use only.

QUESTION	1-10	11	12	13	14	15	**Total**
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I (10 marks)
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section.

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1. A simplified form of $\left(\dfrac{\cos 2\theta + i\sin 2\theta}{\cos\theta - i\sin\theta}\right)$ is

   A. $\cos\theta + i\sin\theta$

   B. $\cos 3\theta + i\sin 3\theta$

   C. $\cos\theta - i\sin\theta$

   D. $\cos 3\theta - i\sin 3\theta$

2. The contrapositive statement to ”If a function is even then its inverse is not even” is:

   A. If the inverse is not even then the function is even.

   B. If the inverse is not even then the function is not even.

   C. If the inverse is even then then the function is even.

   D. If the inverse is even then the function is not even.

3. Which of the following expressions is the partial fraction form of the algebraic fraction $\dfrac{x-1}{(x-2)^2(x^2+3)}$?

   P, Q, R and S are constants,

   A. $\dfrac{P}{(x-2)^2} + \dfrac{Q}{x^2+3}$.

   B. $\dfrac{P}{(x-2)^2} + \dfrac{Qx+R}{x^2+3}$.

   C. $\dfrac{P}{(x-2)} + \dfrac{Q}{(x-2)^2} + \dfrac{R}{x^2+3}$.

   D. $\dfrac{P}{(x-2)} + \dfrac{Q}{(x-2)^2} + \dfrac{Rx+S}{x^2+3}$.

4. z is a complex number such that $z = (1 - a\cos\theta) + i(1 - b\sin\theta)$, where a, b and θ are all real. If the locus of z in an Argand diagram is the straight line $x + y = 1$, which of the following is true?

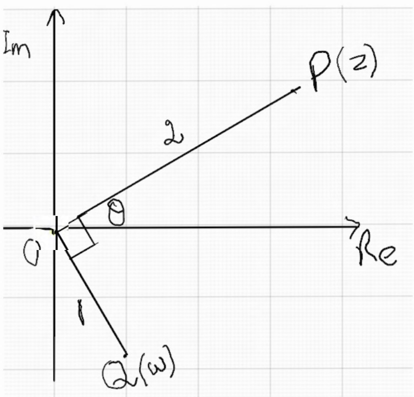
   A. $a^2 + b^2 \leq 1$.

   B. a and b are free to take any real values.

   C. $a^2 + b^2$ must be equal to 1.

   D. The values of a and b are not restricted and there is no relationship between a and b.

5. In the Argand diagram below. P and Q represent the complex numbers z and ω respectively. $\angle POQ = \frac{\pi}{2}$, $|z| = 2$ units and $|\omega| = 1$ unit. Which of the following is correct?



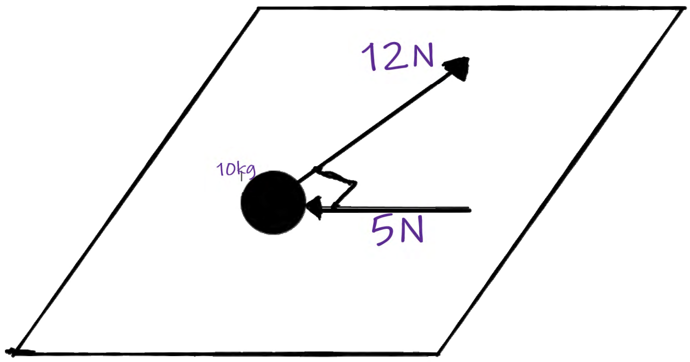
   A. $\omega = -e^{i\left(\frac{\pi}{2}-\theta\right)}$

   B. $\omega = e^{-i\left(\frac{\pi}{2}-\theta\right)}$

   C. $\omega = -e^{-i\left(\frac{\pi}{2}-\theta\right)}$

   D. $\omega = e^{i\left(\frac{\pi}{2}-\theta\right)}$

6. A 10 kg object placed on a smooth horizontal table is subject to two horizontal forces of 12N and 5N in two perpendicular directions as shown.



Calculate the acceleration of the object.

A. $1.7ms^{-2}$

B. $13ms^{-2}$

C. $1.3ms^{-2}$

D. $0.7ms^{-2}$

7. Using a suitable substitution, $\int_{1}^{\sqrt{3}} \frac{\log(\tan^{-1} x)}{1+x^2}\, dx$ can be expressed in terms of u as:

A. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \log u \, du$

B. $\int_{0}^{\frac{\pi}{3}} \log u \, du$

C. $\int_{0}^{\frac{\pi}{2}} \frac{\log(u)}{1+\tan^2 u} \, du$

D. $\int_{0}^{\frac{\pi}{6}} \frac{\log(u)}{1+\tan^2 u} \, du$

8. Let $z = \left(\frac{1+i}{\sqrt{2}}\right)^i$ and $\omega = \left(\frac{1-i}{\sqrt{2}}\right)^i$. Which of the following statement is true about the relationship between z and ω?

A. $z < \omega$.

B. $z > \omega$.

C. complex numbers cannot be ordered. Therefore neither $z < \omega$ nor $z > \omega$ can be true.

D. $|z| = |\omega|$.

9. The polynomial $P(z)$ has the equation $P(z) = z^4 - 2z^3 + \alpha z + 15$, where α is real. Given that $(2 - i)$ is a zero of $P(z)$, which of the following expressions is $P(z)$ as a product of two real quadratic factors?

   A. $(z^2 + 4z + 5)(z^2 + 2z + 3)$

   B. $(z^2 - 4z + 5)(z^2 + 2z + 3)$

   C. $(z^2 + 4z + 5)(z^2 - 2z + 3)$

   D. $(z^2 - 4z + 5)(z^2 - 2z + 3)$

10. The equations of the lines l_1, l_2 and l_3 are give by

    l_1: **r**= 2**i** + 3**j** + 4 **k**+λ(2**i** - 3**j** + 5 **k**)

    l_2: **r**= 2**i**$-$5**j**+4**k** + μ(x**i** + y**j** + z**k**)

    l_3: **r** = $-$3**i** + 2**j** - **k**+ ν (**i** + **j** + $\frac{1}{5}$**k**)

    where λ, μ and ν are constants.The line l_1 is $\parallel$ to l_2 and l_1 is $\perp$ to l_3.

$I.$	$x = -4,$	$y = 6,$	$z = 10$
$II.$	$x = 8,$	$y = -12,$	$z = 20$
$III.$	$x = -4,$	$y = 6,$	$z = -10$

    Which of the following is true?

   A. *I* and *II*.

   B. *I* and *III*

   C. *II* and *III*.

   D. *I*, *II* and *III*

End of Section 1

Section II

90 marks

Attempt Questions 11 to 16 Allow approximately 2 h 45 min for this section
Write your answers on the writing booklet supplied.
Your responses should include relevant mathematical reasoning and/or calculations.
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Your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 Marks)

(a) Solve the equation $z^2 - (1-4i)z - 5 + i = 0$ 3

Give the answer in $a + ib$ form where a and b are real

(b) (i) The complex number z and ω are such that $z = \dfrac{3a - 5i}{1 + 2i}$ and $\omega = 1 + 13bi$, 2

where a and b are real. Given that $\overline{z} = \omega$, find the exact values of a and b.

(ii) On an Argand diagram, sketch the regions representing z 2

where $\text{Re}(z+1) \leq 2$ and $|z - 2i| \leq 2$.

Hence find the greatest value of $|z+3|$ and the complex number z 3

when $|z+3|$ is maximum.

(c) Complex numbers p and q are given by $p = \dfrac{1+i}{1-i}$ and $q = \dfrac{\sqrt{2}}{1-i}$

On an Argand diagram with origin O, Sketch the points P , Q and R representing 2

respectively p , q and $p + q$ respectively.

Hence varify that $arg\left(\dfrac{1+\sqrt{2}+i}{1-i}\right) = \dfrac{3\pi}{8}$ 3

End of Question 11

Question 12. (15 Marks)

(a) Decompose $\dfrac{x^2}{4x^2-9}$ into $A+\dfrac{B}{2x-3}+\dfrac{C}{2x+3}$ where A, B and C are real and hence evaluate: $\displaystyle\int_0^1 \frac{x^2}{4x^2-9}dx$ 4

(b) Using integration by parts to show that: $\displaystyle\int_1^2 \tan^{-1}\sqrt{x^2-1}\ \text{dx}= \frac{2\pi}{3}-\ln 3$ 2

(c) Find $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right)$ 1

Hence find: $\displaystyle\int \frac{1-\ln x}{x\ln x}\,dx$ 2

(d) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{1+\sec\theta} = \frac{\pi}{12}+\sqrt{2}-1-\frac{1}{\sqrt{3}}$ 3

(e) p, q and k are real numbers. If $\displaystyle\int_0^{\frac{\pi}{2}} \sin\left(\frac{p}{p+q}x\right)\cos\left(\frac{q}{p+q}x\right)\,dx = k,$ 3

Find the value of k in terms of p and q.

End of Question 12

Question 13. (15 Marks)

(a) The position vectors of the points A, B and C are $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $4\mathbf{i} + 3\,\mathbf{j} + 5\mathbf{k}$ and $10\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$ respectively.

(i) Show that A, B and C are collinear. 2

(ii) Find the exact length of projection of $\overrightarrow{OA}$ on the line OB. 2

(iii) Relative to the origin O, points A and B have position vectors 2

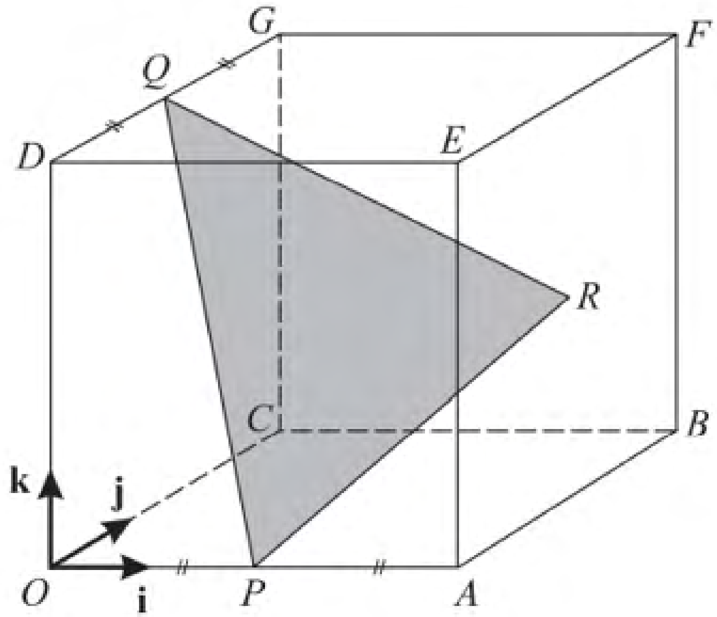
$\begin{bmatrix} c \\ 3 \\ d \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 12 \\ 2 \end{bmatrix}$ respectively, where c and d are constants.

The straight line through A and B has the equation $\mathbf{r} = \begin{bmatrix} 10 \\ 12 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

where λ is a constant. Find the values of c and d.

Question 13 continues on the next page —

(b) The diagram below shows a cube $OABCDEFG$ in which the length of each side is 4 units.The unit vectors **i** , **j** and **k** are parallel to $\overrightarrow{\text{OA}}$, $\overrightarrow{\text{OC}}$ and $\overrightarrow{OD}$ respectively.
The mid-points of OA andDG are P and Q respectively and$R$ is the centre of the square face $ABFE$



(i) Express each of the vectors $\overrightarrow{\text{PR}}$, and $\overrightarrow{\text{PQ}}$ in terms of **i** , **j** and **k**. 2

(ii) Use scalar product to find angle QPR 2

(iii) Find the area of triangle PQR correct to 1 decimal place 2

(c) "If both ab and $a + b$ are even then both a and b are even". 3
Prove this statement by proving its contrapositive.

End of Question 13

Question 14. (15 Marks)

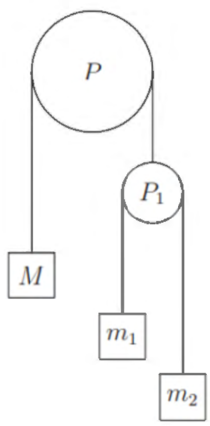
(a) Show that $\dfrac{(x+k)^2}{x^2+x+1} \leq \dfrac{4}{3}(k^2-k+1)$ for all real x and k 3

(b) Use the substitution $x = \cos^2\theta + 4\sin^2\theta$ to show that :

$$\int_1^4 \frac{x}{\sqrt{(x-1)(4-x)}}\,dx = \frac{5\pi}{2}$$ 4

(c) If $I_n = \displaystyle\int_0^1 x^n(1+x^3)^7\,dx$, show that $I_n = \dfrac{256}{n+22} - \dfrac{n-2}{n+22}I_{n-3}$ 3

(d) The diagram below show a system of particles, strings and pulleys. In the system, the pulleys are smooth and light, the strings are light and inextensible, the particles move vertically and the pulley labelled with P is fixed. The masses of the particles are as indicated on the diagrams.



For the pulley system shown above, show that the acceleration, a, of the particle of mass M, measured 5

in the downwards direction, is given by $a = \dfrac{M-4\mu}{M+4\mu}g$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2}$

End of Question 14

Question 15. (15 Marks)

(a) In an aerobatics display, Jane and Karen jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v, Jane experiences air resistance kv per unit mass but Karen, who spread-eagles, experience air resistance $\left(kv + \dfrac{2k^2v^2}{g}\right)$ per unit mass.

(i). Show that Jane's speed can never reach $\frac{g}{k}$ 3

(ii). Obtain the corresponding result for Karen 4

(iii). Jane opens her parachute when her speed is $\dfrac{g}{3k}$. 2

Show that she has been in free fall for time $\frac{1}{k}\ln\frac{3}{2}$

(iv). Karen also opens her parachute when her speed is $\dfrac{g}{3k}$. 2

Find the time she has then been in free fall.

(b) P, Q, R and S are real numbers. Show that if $\dfrac{P}{Q} = \dfrac{R}{S}$, then $\dfrac{P}{Q} = \dfrac{P+R}{Q+S}$ 1

Hence if $\dfrac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \dfrac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = T$, then show that 3

$T = \cos(x+y) + \cos(y+z) + \cos(z+x)$

*E*nd of Question 15

Question 16. (15 Marks)

(a) Given that $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx = \frac{1}{2}a,$ 3

evaluate the integral $\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,dx$

(b) Use a suitable substitution to evaluate $\int_{0.5}^2 \frac{\sin x}{x[\sin x + \sin(\frac{1}{x})]}\,dx$ 3

(c) The cartesian equations of two lines are given by $y = m_1 + c_1$ and $y = m_2 + c_2$.

(i) find the vector equations of the lines 1

(ii) Hence show that the acute angle of the intersection of the lines is given by 2

$tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

(d) (i) Let $y = (x-a)^n e^{bx}\sqrt{1+x^2}$, where n and a are constants and 3
b is a non-zero constant.

Show that $\frac{dy}{dx} = \frac{(x-a)^{n-1}e^{bx}q(x)}{\sqrt{1+x^2}}$, where $q(x)$ is a cubic polynomial.

(ii) Hence determine 3

$$\int \frac{(x-2)^6 e^{4x}(4x^4+x^3-2)}{\sqrt{1+x^2}}dx$$

—End of Paper—-

NSBHS Mathematics Ext 2 - Trials 2021 (Task 3)

Sample Solutions

1. B

2. D

3. D

4. A

5. B

6. C

7. A

8. A, C

9. B

10. C

(a) $z^2 - (1-4i)z - 5 + i = 0$

$$z = \frac{(1-4i) \pm \sqrt{(1-4i)^2 + 4(5-i)}}{2}$$

$$= \frac{(1-4i) \pm \sqrt{5-12i}}{2}$$

$$= \frac{(1-4i) \pm (3-2i)}{2}$$

$$= (2-3i) \text{ or } (-1-i)$$

(b) (i) $z = \dfrac{3a-5i}{1+2i}$, $w = 1+13bi$

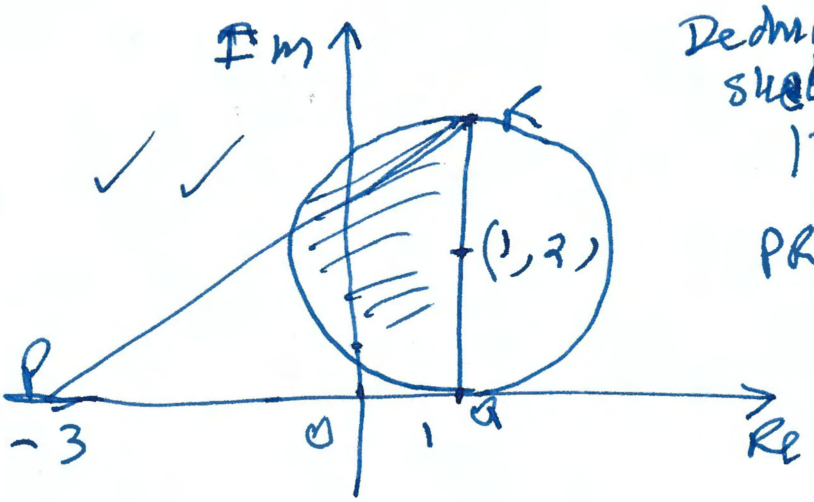
$$\bar{z} = w \Rightarrow \frac{3a+5i}{1-2i} = 1+13bi$$

$$\Rightarrow 3a+5i = (1+26b) + (13b-2)i$$

Matching the real and imaginary parts both sides,

$$\left.\begin{array}{l} 26b+1 = 3a \\ \text{and } 13b-2 = 5 \end{array}\right\} \Rightarrow a = 5,\ b = \frac{7}{13}$$

(ii) $\left\{\begin{array}{l} Re(z+1) \le 2 \equiv \text{region left of the line } Re\,z = 1 \\ |z-1-2i| \le 2 \equiv \text{circular region centred at (} \end{array}\right.$



Deduce from the sketch that

$|z+3|_{max} = PR$

$PR = \sqrt{PQ^2 + QR^2}$

$= \sqrt{16+16}$

$= 4\sqrt{2}$

when z is at R,

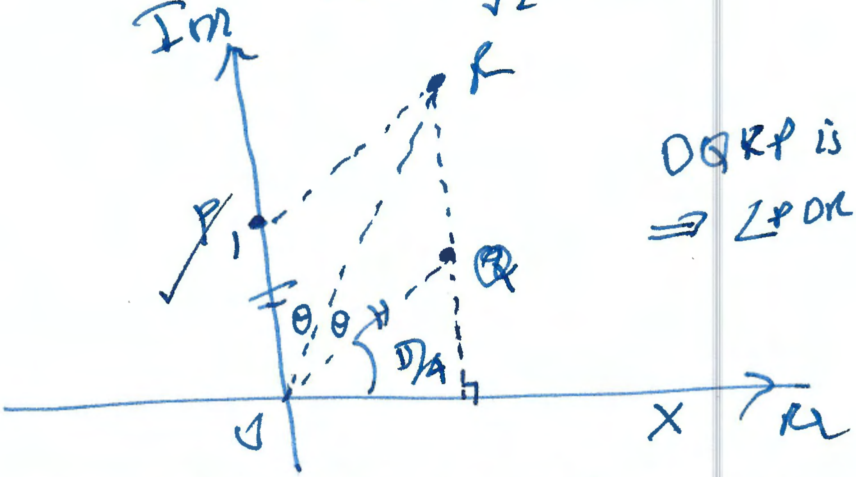
$z = 1+4i$

$p = \frac{1+i}{1-i} \Rightarrow \arg p = \frac{\pi}{4} - (-\pi/4) = \frac{\pi}{2}$

$|p| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$q = \frac{\sqrt{2}}{1-i} \Rightarrow \arg q = 0 - (-\pi/4) = \pi/4$

$|q| = \frac{\sqrt{2}}{\sqrt{2}} = 1$



OQRP is a rhombus

$\Rightarrow \angle POR = \angle ROQ = \theta$

Deduce that $\frac{1+\sqrt{2}+i}{1-i} = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = p + q$ represents by R

$\arg\left(\frac{1+\sqrt{2}+i}{1-i}\right) = \angle ROX = \angle ROQ + \angle QOX$

$= \theta + \pi/4$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \pi/4$

$= \frac{3\pi}{8}$

(a)
$$\frac{x^2}{4x^2-9} = A + \frac{B}{2x-3} + \frac{C}{2x+3}$$

$$\Rightarrow A = \frac{1}{4},\ B = \frac{3}{8},\ C = -\frac{3}{8}$$

$$\therefore \int_0^1 \frac{x^2}{4x^2-9}\,dx = \int_0^1 \frac{1}{4} + \frac{3/8}{2x-3} - \frac{3/8}{2x+3}\,dx$$

$$= \left[\frac{x}{4} + \frac{3}{16}\ln\left|\frac{2x-3}{2x+3}\right|\right]_0^1$$

$$= \frac{1}{4} + \frac{3}{16}\ln\left(\frac{1}{5}\right)$$

(b)
$$\int_1^2 \tan^{-1}(\sqrt{x^2-1})\,dx = \left[x\tan^{-1}\sqrt{x^2-1} - \int \frac{1}{\sqrt{x^2-1}}\,dx\right]_1^2$$

$$= \left(x\tan^{-1}\sqrt{x^2-1} - \ln\left[x+\sqrt{x^2-1}\right]\right)_1^2$$

$$= \frac{2\pi}{3} - \ln(2+\sqrt{3})$$

$\boxed{\text{* } \int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx \text{ can be evaluated using the sub } x = \sec\theta.}$

(c)
$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2}$$

$$\int \frac{1-\ln x}{x\ln x}\,dx = \int \frac{1-\ln x}{x^2}\cdot\frac{x}{\ln x}\,dx$$

$$= \int \frac{d}{dx}\left(\frac{\ln x}{x}\right)\cdot\frac{1}{\left(\frac{\ln x}{x}\right)}\,dx$$

$$= \int \frac{1}{\left(\frac{\ln x}{x}\right)}\,d\left(\frac{\ln x}{x}\right)$$

$$= \ln\left(\frac{\ln x}{x}\right) + C$$

1 b (d)

$$\int_{\pi/4}^{\pi/3} \frac{d\theta}{1+\sec\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta - 1}{\tan^2\theta} \, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cot\theta \operatorname{cosec}\theta - \operatorname{cosec}^2\theta + 1 \, d\theta$$

$$= \left[-\operatorname{cosec}\theta + \cot\theta + \theta\right]_{\pi/4}^{\pi/3}$$

$$= \left(-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{\pi}{3}\right) - \left(-\sqrt{2} + 1 + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$$

(e)

$$\int_0^{\pi/2} \sin\left(\frac{p}{p+q}\right)x \cdot \sin\left(\frac{q}{p+q}\right)x \, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\left(\frac{p-q}{p+q}\right)x - \cos x \, dx$$

$$= \frac{1}{2}\left[\frac{p+q}{p-q}\sin\left(\frac{p-q}{p+q}\right)x - \sin x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{p+q}{p-q}\right)\sin\left(\frac{p-q}{p+q}\right)\cdot\frac{\pi}{2} - 1\right] = k$$

1) (i) $\vec{AB} = (4\underset{\sim}{i}+3\underset{\sim}{j}+5\underset{\sim}{k}) - (\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k})$

$= 3\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k}.$

$\vec{AC} = (10\underset{\sim}{i}+5\underset{\sim}{j}+9\underset{\sim}{k}) - (\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k})$

$= 9\underset{\sim}{i}+3\underset{\sim}{j}+6\underset{\sim}{k}$

$= 3(3\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k})$

$= 3\vec{AB}.$

$\therefore$ A, B and C are collinear.

(ii) The length of the projection of $\vec{OA}$ onto $\vec{OB}$

$= \left|\text{proj}_{\vec{OB}}\vec{OA}\right| = \left||\vec{OA}|\cos\theta\right|$

$= \dfrac{\vec{OA}\cdot\vec{OB}}{|\vec{OB}|}$

$= \dfrac{(4+6+15)}{\sqrt{4^2+3^2+5^2}}$

$= \dfrac{25}{\sqrt{50}}$

$= \dfrac{5}{\sqrt{2}}.$

(iii) $\begin{bmatrix}10\\12\\2\end{bmatrix} + \lambda\begin{bmatrix}1\\3\\0\end{bmatrix} = \begin{bmatrix}c\\3\\d\end{bmatrix}$

$\Rightarrow$ $10+\lambda = c$
$12+3\lambda = 3 \Longrightarrow \lambda = -3$
$2 = d$

$\therefore \left.\begin{array}{l} c = 7 \\ d = 2 \end{array}\right\}$

1 (b)

(i) $\underset{\sim}{p} = 2\underset{\sim}{i}$

$\underset{\sim}{q} = 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{r} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{PR} = \underset{\sim}{r} - \underset{\sim}{p} = (2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p} = (-2\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$

(ii) $\cos \angle QPR = \dfrac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{|\overrightarrow{PR}| \cdot |\overrightarrow{PQ}|}$

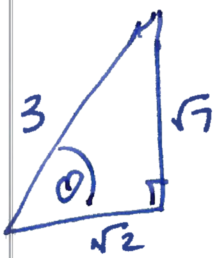
$= \dfrac{-4 + 4 + 8}{\sqrt{2^2 + 2^2 + 2^2} \cdot \sqrt{2^2 + 2^2 + 4^2}}$

$= \dfrac{8}{12\sqrt{2}}$

$= \dfrac{\sqrt{2}}{3}$

$\therefore \angle QPR =$

(iii) $\sin \angle QPR = \dfrac{\sqrt{7}}{3}$



Area of $\Delta PQR = \frac{1}{2} |PR| |PQ| \sin \angle QPR$

$= \frac{1}{2} \sqrt{12} \sqrt{24} \cdot \frac{\sqrt{7}}{3}$

$= 2\sqrt{14}$ sq. units

13(c)

We can prove the statement by proving its contrapositive:

"If a and b are not both even, then ab and (a+b) are not both even"

(ie) If either a or b is odd, then either ab or a+b is odd

Suppose a is odd. Then a = 2k for some integer k.

Then ab = (2k+1)b = 2kb + b which is odd if b is odd, even otherwise

And a+b = 2k+1+b which is odd if b is even, and even otherwise.

So regardless of whether b is even or odd, (it must be one or the other), one of (a+b) or ab is necessarily odd.

The argument is identical if we suppose b is odd.

2 (a) let $y = \dfrac{(x+k)^2}{x^2+x+1}$

Rearranging it gives,

$(y-1)x^2 + (y-2k)x + (y-k^2) = 0$

This is a quadratic equation in x.

$\therefore \forall$ real x, $\Delta \geq 0$

$\Rightarrow (y-2k)^2 - 4(y-1)(y-k^2) \geq 0$

$\Rightarrow 3y^2 - 4(k^2-k+1)y \leq 0$

$\Rightarrow y \leq \frac{4}{3}(k^2-k+1)$

(b) $x = \cos^2\theta + 4\sin^2\theta$

$\frac{dx}{d\theta} = -2\cos\theta\sin\theta + 8\sin\theta\cos\theta$

$= 3\sin 2\theta$

$x=1$ $\quad 1 = \cos^2\theta + 4\sin^2\theta \Rightarrow \sin^2\theta = 0 \Rightarrow \theta = 0$

$x=4$ $\quad 4 = \cos^2\theta + 4\sin^2\theta \Rightarrow \cos^2\theta = 0 \Rightarrow \theta = \pi/2$

$\Rightarrow \int_1^4 \frac{x}{\sqrt{(x-1)(4-x)}}\,dx = \int_0^{\pi/2} \frac{\cos^2\theta + 4\sin^2\theta}{\sqrt{3\sin^2\theta \cdot 3\cos^2\theta}} \cdot 3\sin 2\theta \, d\theta$

$= \int_0^{\pi/2} 2\cos^2\theta + 8\sin^2\theta \, d\theta$

$= \int_0^{\pi/2} (5 - 3\cos 2\theta)\, d\theta$

$= \left[5\theta - \frac{3}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \frac{5\pi}{2}$ //

14(c)

$$I_n = \int_0^1 x^n (1+x^3)^7 dx$$

$$= \int_0^1 \frac{x^{n-2}}{3} \cdot 3x^2 (1+x^3)^7 dx$$

$$= \left[\frac{x^{n-2}(1+x^3)^8}{24} - \int \frac{(1+x^3)^8}{24} \cdot (n-2) x^{n-3} dx \right]_0^1$$

$$= \cancel{\frac{256}{24}}$$

$$= \left[\frac{256}{24}\right] - \frac{(n-2)}{24} \int_0^1 x^{n-3} \cdot (1+x^3)^8 dx$$

$$= \frac{256}{24} - \frac{(n-2)}{24} \int_0^1 x^{n-3} (1+x^3)(1+x^3)^7 dx$$

$$= \frac{256}{24} - \frac{(n-2)}{24} \left[\int_0^1 x^{n-3} (1+x^3)^7 + x^n (1+x^3)^7 dx \right]$$

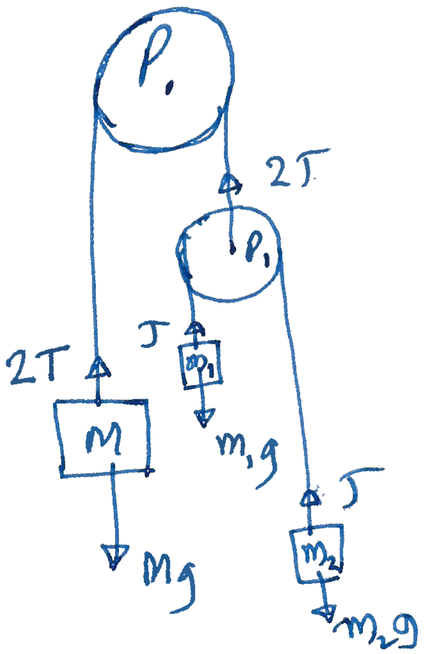
$$= \frac{256}{24} - \frac{(n-2)}{24} \left[I_{n-3} + I_n \right]$$

$$I_n = \frac{257}{24} - \frac{(n-2)}{24} \left[I_{n-3} \right.$$

$$I_n \left[1 + \frac{n-2}{24} \right] = \frac{256}{24} - \frac{(n-2)}{24} I_{n-3}$$

$$\Rightarrow \quad I_n = \frac{256}{(n+22)} - \frac{(n-2)}{(n+22)} I_{n-3}$$

(ii)



Let M moves downward with acceleration a and m_2 moves downward with respect to P_1 with acceleration a_1

Applying Newton's 2nd law on:

M ↓ $Mg - 2T = Ma$ —— (1)

m_1 ↑ $T - m_1 g = m_1(a + a_1)$ —— (2)

m_2 ↑ $T - m_2 g = m_2(a - a_1)$ —— (3)

$m_2 \times (2) + m_1 \times (3)$

$\Rightarrow (m_1 + m_2)T - 2m_1 m_2 g = 2m_1 m_2 a$ —— (4)

$(m_1 + m_2) \times (1) + 2 \times (4)$

$$M(m_1+m_2)g - 4m_1 m_2 g = [M(m_1+m_2) + 2m_1 m_2]a$$

$$\therefore a = \frac{[M(m_1+m_2) - 4m_1 m_2]g}{M(m_1+m_2) + 2m_1 m_2}$$

Dividing ~~by~~ all by $(m_1 + m_2)$

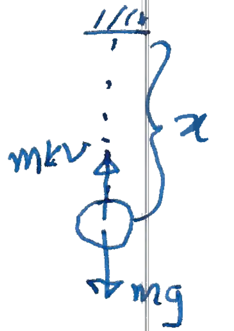
$$a = \frac{\left[M - 4\frac{m_1 m_2}{m_1+m_2}\right]g}{\left[M + \frac{2m_1 m_2}{m_1+m_2}\right]}$$

$$= \frac{(M - 4\mu)g}{M + 2\mu}$$

When $\mu = \frac{m_1 m_2}{m_1 + m_2}$

(i) Jane

Applying $\downarrow F = ma$



$\downarrow \quad mg - mkv = m\ddot{x}$

$\ddot{x} = g - kv$

$\frac{dv}{dt} = g - kv$

$\int_0^v \frac{dv}{g-kv} = \int_0^t dt$

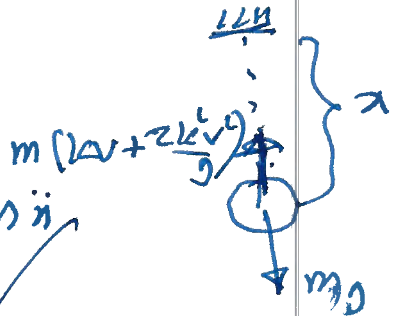
$-\frac{1}{k}\ln\left(1 - \frac{kv}{g}\right) = t$

For finite values of t, $\frac{kv}{g} - 1 \neq 0$

$\Longrightarrow v \neq \frac{g}{k}$

(ii) Karen

Applying $\downarrow F = ma$



$\downarrow \quad mg - m\left(kv + \frac{2k^2v^2}{g}\right) = m\ddot{x}$

$\ddot{x} = g - kv - \frac{2k^2v^2}{g}$

$= -\frac{1}{g}\left[2k^2v^2 + kgv - g^2\right]$

$\ddot{x} = -\frac{1}{g}\left[(2kv - g)(kv + g)\right]$

$\int_0^v \frac{-g\,dv}{(2kv-g)(kv+g)} = \int_0^t dt$

$\int_0^v \left(\frac{-2/3}{(2kv-g)} + \frac{1/3}{g+kv}\right) dv = \int_0^t dt$

$\frac{1}{3k}\ln\left|\frac{g+kv}{2kv-g}\right| = t$

For finite values of t, $2kv - g \neq 0$

$\rightarrow v \neq \frac{g}{2k}$

15(a) (iii) When Jane's speed $= \frac{g}{3k}$,

$$-\frac{1}{k}\ln\left(1-\frac{1}{3}\right) = t$$

$$\cancel{t = \frac{1}{k}\ln\left(\frac{3}{2}\right)}$$

$$\Rightarrow t = \frac{1}{k}\ln\left(\frac{3}{2}\right)$$

$\therefore$ Jane has been in free fall for $\underline{\underline{\frac{1}{k}\ln\left(\frac{3}{2}\right)}}$ seconds

15(a)(iv) When Karen's speed $= \frac{g}{3k}$,

$$\frac{1}{3k}\ln\left|\frac{4/3}{1/3}\right| = t$$

$$\Rightarrow t = \frac{1}{3k}\ln 4$$

$\therefore$ Karen has been in free fall for $\underline{\underline{\frac{1}{3k}\ln 4}}$ $\underline{\underline{\text{seconds}}}$

15(b) (i) Let $\frac{P}{Q} = \frac{iR}{iS} = k$

$$\Rightarrow P = Qk$$

$$iR = iSk$$

$$P + iR = (Q + iS)k$$

$$\Rightarrow k = \frac{P + iR}{Q + iS}$$

$$\Rightarrow \frac{P}{Q} = \frac{P + iR}{Q + iS}$$

5/1 b)

$$\frac{P}{Q} = \frac{R}{S} = \frac{P+R}{Q+S}$$

$$= \frac{iR}{iS} = \frac{P+iR}{Q+iS}$$

let $P = \cos x + \cos y + \cos z$

$Q = \cos(x+y+z)$

$R = \sin x + \sin y + \sin z$

$S = \sin(x+y+z)$

$$\cancel{\frac{P}{Q} = \frac{iR}{iS}}$$

$$\frac{P}{Q} = \frac{P+iR}{Q+iS} = T.$$

$$\Rightarrow \quad T = \frac{(\cos x + \cos y + \cos z) + i(\sin x + \sin y + \sin z)}{\cos(x+y+z) + i\sin(x+y+z)}$$

$$= \frac{e^{ix} + e^{iy} + e^{iz}}{e^{i(x+y+z)}}$$

$$= e^{-i(y+x)} + e^{-i(z+x)} + e^{-i(y+z)}$$

But T is Real

$$\Rightarrow \quad T = \mathrm{Re}\left[e^{-i(x+y)} + e^{-i(y+z)} + e^{-i(z+x)}\right]$$

$$= \cos(x+y) + \cos(y+z) + \cos(z+x)$$

/b/ (a) $\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)} dx$

$= \int_0^1 \frac{\ln(x+1)}{\ln[(x+1)(-x+2)]} dx$ ✓

$= \int_0^1 \frac{\ln(x+1)}{\ln(x+1)+\ln(2-x)} dx$

$= \int_0^1 \frac{\ln(x+1)}{\ln(x+1)+\ln[1-(x-1)]} dx$ ✓

$\left[\begin{array}{l} f(x) = \ln(x+1) \\ f(1-x) = \ln(1-x+1) = \ln(2-x) \end{array}\right]$

✓

$= \frac{1}{2}$ //

(b) let $I = \int_{0.5}^{2} \frac{\sin x}{x[\sin x + \sin(\frac{1}{x})]} dx$

let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$ ✓

$x = 0.5,\ u = 2$

$x = 2,\ u = 0.5$

$\Rightarrow I = \int_2^{0.5} \frac{\sin(\frac{1}{u})}{(\cancel{x})[\sin(\frac{1}{u}) + \sin(u)]} \cdot -x^2 du$ ✓

$= \int_{0.5}^{2} \cancel{x} \frac{\sin(\frac{1}{u})\ du}{u[\sin(\frac{1}{u}) + \sin u]}$

$\Rightarrow \cancel{2I = \int_{0.5}^{2} \frac{1}{x} dx} = \int_{0.5}^{2} \frac{\sin(\frac{1}{x})}{x[\sin x + \sin(\frac{1}{x})]} dx$

$\Rightarrow 2I = \int_{0.5}^{2} \frac{1}{x} dx = [\ln x]_{0.5}^{2} = \ln 4 = 2\ln 2$

$\therefore I = \ln 2$ ✓

Q16 (b)

(i) $y = m_1 x + c_1$ It passes through $(0, c_1)$ and gradient m_1

$\therefore$ The vector equation is

$$\underset{\sim}{r} = c_1 \underset{\sim}{j} + \lambda(\underset{\sim}{i} + m_1 \underset{\sim}{j}) \quad (\lambda \in \mathbb{R})$$

Similarly the vector equation of $y = m_2 x + c_2$ is

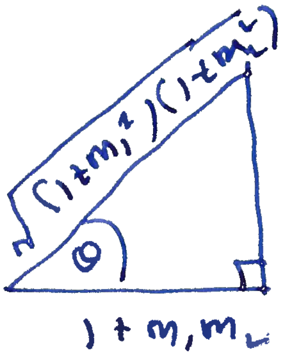
$$\underset{\sim}{r} = c_2 \underset{\sim}{j} + \mu(\underset{\sim}{i} + m_2 \underset{\sim}{j}) \quad (\mu \in \mathbb{R})$$

(ii) If θ is the acute angle between them

then $(\underset{\sim}{i} + m_1 \underset{\sim}{j}) \cdot (\underset{\sim}{i} + m_2 \underset{\sim}{j}) = |\underset{\sim}{i} + m_1 \underset{\sim}{j}||\underset{\sim}{i} + m_2 \underset{\sim}{j}| \cos\theta$

$$\Rightarrow \quad 1 + m_1 m_2 = \sqrt{1+m_1^2}\,\sqrt{1+m_2^2} \cdot \cos\theta$$

$$\cos\theta = \frac{1 + m_1 m_2}{\sqrt{1+m_1^2} \cdot \sqrt{1+m_2^2}}$$



$$\tan\theta = \frac{\sqrt{(1+m_1^2)(1+m_2^2) - (1+m_1 m_2)^2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{m_1^2 - 2m_1 m_2 + m_2^2}}{1 + m_1 m_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

16(d) (i) $y = (x-a)^n e^{bx}\sqrt{1+x^2}$

$$\frac{dy}{dx} = n(x-a)^{n-1}e^{bx}\sqrt{1+x^2} + (x-a)^n\left[b e^{bx}\sqrt{1+x^2} + e^{bx}\cdot\frac{x}{\sqrt{1+x^2}}\right]$$

$$= (x-a)^{n-1}e^{bx}\left[n\sqrt{1+x^2} + \frac{(x-a)\left[b(1+x^2)+x\right]}{\sqrt{1+x^2}}\right]$$

$$= \frac{(x-a)^{n-1}e^{bx}}{\sqrt{1+x^2}}\left[n(1+x^2) + b(x-a)(1+x^2) + x(x-a)\right]$$

$$= \frac{(x-a)^{n-1}e^{bx}}{\sqrt{1+x^2}}\left[bx^3 + (n-ab+1)x^2 + (b-a)x + (n-ab)\right]$$

$$= \frac{(x-a)^{n-1}\cdot e^{bx}}{\sqrt{1+x^2}}\cdot q(x) \quad \text{where } q(x) \text{ is a cubic polynomial}$$

(ii) $n=8,\ a=2,\ b=4$

$$(1)\quad \int \frac{(x-2)^7 e^{4x}}{\sqrt{1+x^2}}\cdot(4x^3 + x^2 + 2x)\,dx = (x-2)^8 e^{4x}\sqrt{1+x^2}$$

$n=7,\ a=2,\ b=4$

$$(2)\quad \int \frac{(x-2)^6 e^{4x}}{\sqrt{1+x^2}}(4x^3 + 2x - 1)\,dx = (x-2)^7 e^{4x}\sqrt{1+x^2}$$

(1) + 2 × (2)

$$\int \frac{(x-2)^6 e^{4x}}{\sqrt{1+x^2}}\left[(x-2)(4x^3+x^2+2x) + 2(4x^3+2x-1)\right]dx$$

$$= (x-2)^7 e^{4x}\sqrt{1+x^2}\left[(x-2)+2\right]$$

$$\Rightarrow \int \frac{(x-2)^6 e^{4x}}{\sqrt{1+x^2}}\left[4x^4 + x^3 - 2\right]dx = x(x-2)^7 e^{4x}\sqrt{1+x^2} + C$$